Scale-Discretized Wavelets on the Sphere

Wavelets and Sparsity on the Sphere

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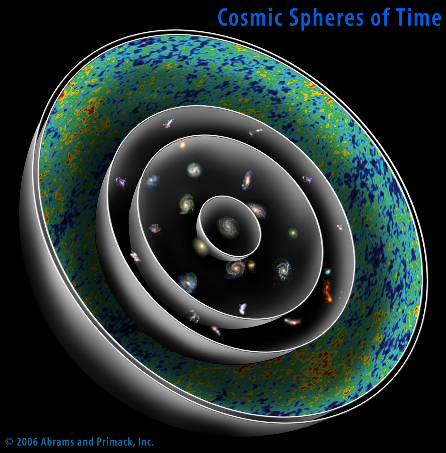
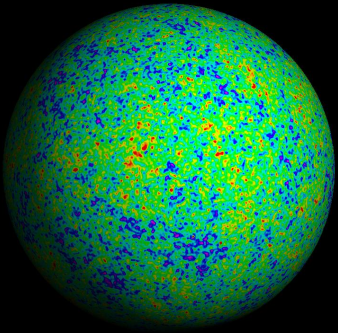
Observations on spherical manifolds
Cosmology

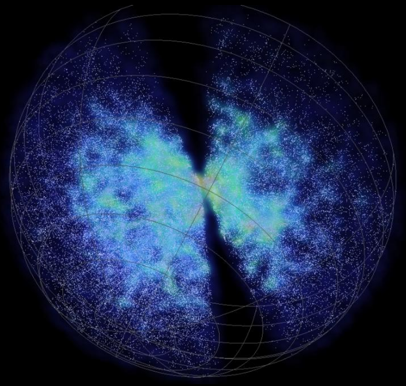


Cosmic microwave background (CMB)



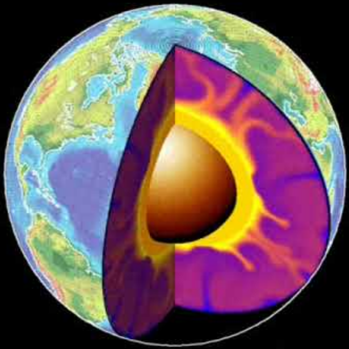
Credit: WMAP

Galaxy surveys



Credit: SDSS

Observations on spherical manifolds
Geophysics



Credit: http://maps.unomaha.edu/

Outline

1. Continuous wavelets on the sphere
   - Stereographic projection
   - Harmonic dilation
2. Scale-discretised wavelets
   - Analysis and synthesis
   - Steerability
   - Exact and efficient computation
3. Future Extensions
   - Exploiting a new sampling theorem on the sphere

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Continuous wavelets on the sphere via stereographic projection

- One of the first natural wavelet construction on the sphere was derived in the seminal work of Antoine and Vandergheynst (1998) (reintroduced by Wiaux 2005).
- Construct wavelet atoms from affine transformations (dilation, translation) on the sphere of a mother wavelet.
- The natural extension of translations to the sphere are rotations. Rotation of a function f on the sphere is defined by

$$[\mathcal{R}(\rho)f](\omega) = f(\rho^{-1} \cdot \omega), \quad \omega = (\theta, \varphi) \in \mathbb{S}^2, \quad \rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3) \ .$$

- How define dilation on the sphere?
- The spherical dilation operator is defined through the conjugation of the Euclidean dilation and stereographic projection Π:

$$\mathcal{D}(a) \equiv \Pi^{-1} \, d(a) \, \Pi \ .$$

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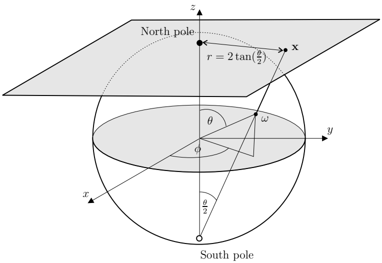

Figure: Stereographic projection.

Continuous wavelet analysis

- Wavelet family on the sphere constructed from rotations and dilations of a mother spherical wavelet Ψ:

$$\{\Psi_{a,\rho} \equiv \mathcal{R}(\rho)\mathcal{D}(a)\Psi : \rho \in \mathrm{SO}(3), a \in \mathbb{R}_*^+\}.$$

- The forward wavelet transform is given by

$$W_\Psi^f(a,\rho) = \langle f, \Psi_{a,\rho} \rangle = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)\, f(\omega)\, \Psi_{a,\rho}^*(\omega)\ ,$$

where $\mathrm{d}\Omega(\omega) = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the usual invariant measure on the sphere.

- Wavelet coefficients live in $\mathrm{SO}(3) \times \mathbb{R}_*^+$; thus, directional structure is naturally incorporated.

- Fast algorithms essential (for a review see Wiaux, McEwen & Vielva 2007)
  - Factoring of rotations: McEwen *et al.* (2007), Wandelt & Gorski (2001), Risbo (1996)
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- FastCSWT code available to download: http://www.jasonmcewen.org/

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Continuous wavelet synthesis

- The inverse wavelet transform is given by

$$f(\omega) = \int_0^\infty \frac{\mathrm{d}a}{a^3} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W_\Psi^f(a, \rho) \, [\mathcal{R}(\rho)\widehat{L}_\Psi \Psi_a](\omega) \, ,$$

where $\mathrm{d}\varrho(\rho) = \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma$ is the invariant measure on the rotation group $\mathrm{SO}(3)$.

- Perfect reconstruction is ensured provided wavelets satisfy the admissibility property:

$$0 < \widehat{C}_\Psi^\ell \equiv \frac{8\pi^2}{2\ell+1} \sum_{m=-\ell}^{\ell} \int_0^\infty \frac{\mathrm{d}a}{a^3} \mid (\Psi_a)_{\ell m} \mid^2 < \infty, \quad \forall \ell \in \mathbb{N}$$

where $(\Psi_a)_{\ell m}$ are the spherical harmonic coefficients of $\Psi_a(\omega)$.

- Continuous wavelets used in many cosmological studies, for example:
  - Non-Gaussianity (*e.g.* Vielva *et al.* 2004; McEwen *et al.* 2005, 2006, 2008)
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- BUT... **exact reconstruction not feasible in practice!**

Continuous wavelets on the sphere via harmonic dilation

- Define dilation by scaling in harmonic space (McEwen *et al.* 2006, Sanz *et al.* 2006):

$$\Psi_{\ell m}(a) = \sqrt{\frac{2\ell+1}{8\pi^2}}\,\Upsilon_m(\ell a)\ ,$$

- Wavelet analysis and synthesis defined in the same manner as stereographic wavelets.
- Admissibility condition defined on the wavelet generating functions Υ

$$0 < C_\Upsilon^\ell = \sum_{m=-\ell}^{\ell} \int_0^\infty \frac{dq}{q}\,|\Upsilon_m(q)|^2 < \infty\ .$$

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Scale-discretised wavelets on the sphere

- **Exact reconstruction not feasible in practice with continuous wavelets!**
- Wiaux, McEwen, Vandergheynst, Blanc (2008)
  Exact reconstruction with directional wavelets on the sphere
- Alternatives: isotropic wavelets, pyramidal wavelets, ridgelets, curvelets (Starck *et al.* 2006); needlets (Narcowich *et al.* 2006, Baldi *et al.* 2009, Marinucci *et al.* 2008)

- Dilation performed in harmonic space
  cf. McEwen *et al.* (2006), Sanz *et al.* (2006).
- The scale-discretised wavelet $\Psi \in \mathrm{L}^2(\mathrm{S}^2, \mathrm{d}\Omega)$ is defined in harmonic space:

$$\Psi^j_{\ell m} \equiv \kappa^j(\ell) s_{\ell m} \: ,$$

- Construct wavelets to satisfy a resolution of the identity:

$$|\Phi_{\ell 0}|^2 + \sum_{j=0}^{J} \sum_{m=-\ell}^{\ell} |\Psi^j_{\ell m}|^2 = 1 \: , \quad \forall \ell \: .$$

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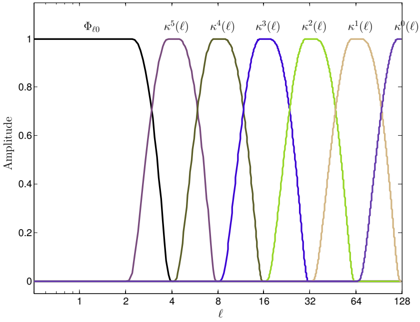

Figure: Harmonic tiling on the sphere.

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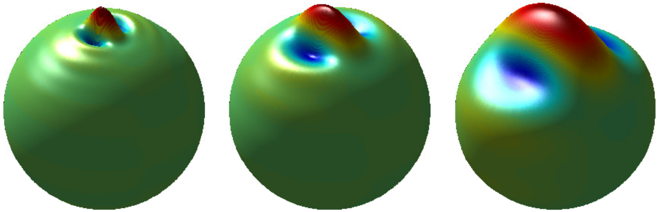
Scale-discretised wavelets on the sphere



(a) $j = 4$ (b) $j = 3$ (c) $j = 2$

Figure: Scale-discretised wavelets on the sphere.

- The scale-discretised wavelet transform is given by the usual projection onto each wavelet:

$$W^{\Psi^j}(\rho) \equiv (f \star \Psi^j)(\rho) = \langle f,\, \mathcal{R}_\rho \Psi^j \rangle = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) f(\omega) (\mathcal{R}_\rho \Psi^j)^*(\omega) \ ,$$

- The original function may be recovered exactly in practice from the wavelet (and scaling) coefficients:

$$f(\omega) = 2\pi \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega') W^{\Phi}(\omega') (\mathcal{R}_{\omega'} L^{\mathrm{d}} \Phi)(\omega) + \sum_{j=0}^{J} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{\Psi^j}(\rho) (\mathcal{R}_\rho L^{\mathrm{d}} \Psi^j)(\omega) \ .$$

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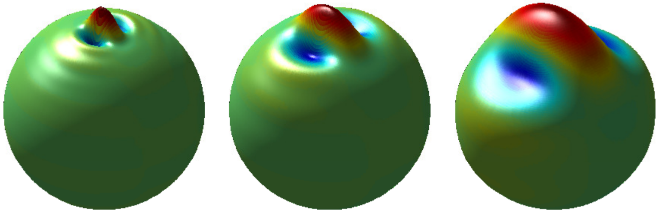

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Steerability

- The scale-discretised wavelet $\Psi \in \mathrm{L}^2(\mathbb{S}^2)$ is defined in harmonic space in factorised form:

$$\Psi^j_{\ell m} \equiv \kappa^j(\ell)\, s_{\ell m} \ .$$

- Without loss of generality, impose

$$\sum_{|m|\leq\ell} |s_{\ell m}|^2 = 1 \ ,$$

such that localisation governed largely by the kernel κ^j and directionality by $s_{\ell m}$.

- By imposing an azimuthal band-limit N, *i.e.* $s_{\ell m} = 0, \ \forall m \geq N$, we recover steerable wavelets:

$$s_\gamma(\omega) = \sum_{g=0}^{M-1} z(\gamma - \gamma_g) s_{\gamma_g}(\omega) \ .$$

- By the linearity of the wavelet transform, steerability extends to wavelet coefficients:

$$W^{\Psi^j}(\alpha, \beta, \gamma) = \sum_{g=0}^{M-1} z(\gamma - \gamma_g) W^{\Psi^j}(\alpha, \beta, \gamma_g) \ .$$

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Exact and efficient computation

- Wavelet analysis can be posed as an inverse Wigner transform on $\mathrm{SO}(3)$:

$$W^{\Psi^j}(\rho) = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \frac{2\ell+1}{8\pi^2} \left(W^{\Psi^j}\right)^{\ell}_{mn} D^{\ell *}_{mn}(\rho) \ ,$$

where

$$\left(W^{\Psi^j}\right)^{\ell}_{mn} = \frac{8\pi^2}{2\ell+1} f_{\ell m} \Psi^{j*}_{\ell n} \ .$$

which can be computed efficiently via a factoring of rotations (Risbo 1996, Wandelt & Gorski 2001).

- Wavelet synthesis can be posed as an forward Wigner transform on $\mathrm{SO}(3)$:

$$f(\omega) \sim \sum_{j=0}^{J} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{\Psi^j}(\rho) (\mathcal{R}_\rho L^{\mathrm{d}} \Psi^j)(\omega) = \sum_{j=0}^{J} \sum_{\ell m n} \frac{2\ell+1}{8\pi^2} \left(W^{\Psi^j}\right)^{\ell}_{mn} \Psi^j_{\ell n} Y_{\ell m}(\omega) \ ,$$

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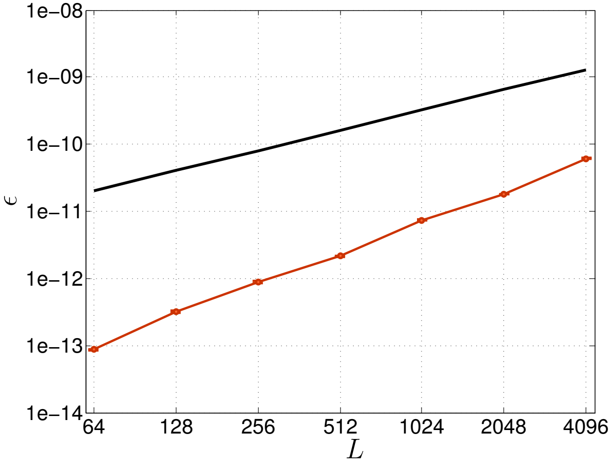


Figure: Numerical accuracy of the scale-discretised wavelet transform.

Exact and efficient computation

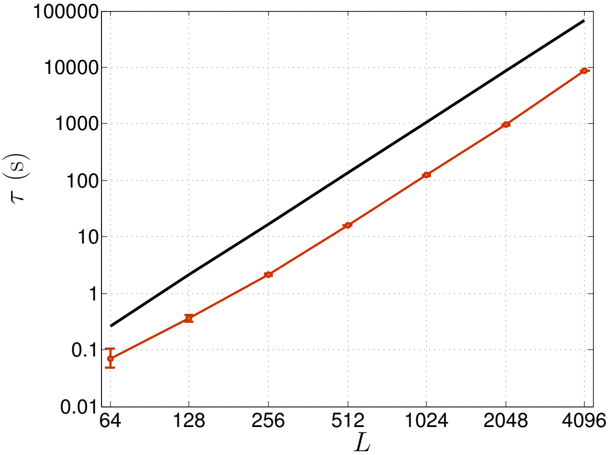


Figure: Computation time of the scale-discretised wavelet transform.

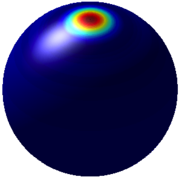
Codes to compute scale-discretised wavelets on the sphere

S2DW code

http://www.s2dw.org

Exact reconstruction with directional wavelets on the sphere
Wiaux, McEwen, Vandergheynst, Blanc (2008)

- Fortran
- Parallelised
- Supports directional, steerable wavelets



S2LET code

http://www.s2let.org

S2LET: A code to perform fast wavelet analysis on the sphere
Leistedt, McEwen, Vandergheynst, Wiaux (2012)

- C, Matlab, IDL, Java
- Supports only axisymmetric wavelets at present
- Future extensions:
  - Directional, steerable wavelets
  - Faster algorithms to perform wavelet transforms
  - Spin wavelets

Outline

Driscoll & Healy (DH) sampling theorem

- Canonical sampling theorem on the sphere derived by Driscoll & Healy (1994).

$\Rightarrow$ $N_{\mathrm{DH}} = (2L-1)2L+1 \sim 4L^2$ samples on the sphere.

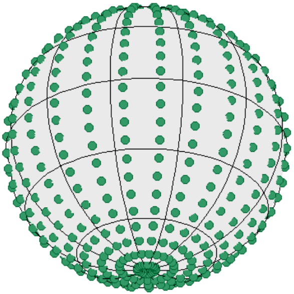


Figure: Sample positions of the DH sampling theorem.

McEwen & Wiaux (MW) sampling theorem

- A new sampling theorem on the sphere (McEwen & Wiaux 2011).

  $\Rightarrow$ $N_{\mathrm{MW}} = (L-1)(2L-1)+1 \sim 2L^2$ samples on the sphere.

- Reduced the Nyquist rate on the sphere by a factor of two.

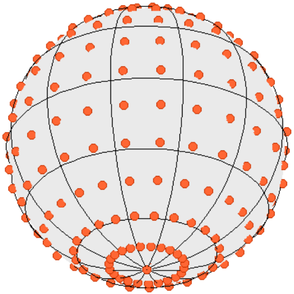


Figure: Sample positions of the MW sampling theorem.

McEwen & Wiaux (MW) sampling theorem

- New sampling theorem follows by associating the sphere with the torus through a periodic extension.
- Similar in flavour to making a periodic extension in θ of a function f on the sphere.

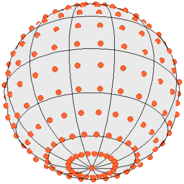
(a) Function on sphere (b) Even function on torus (c) Odd function on torus

Figure: Associating functions on the sphere and torus

Comparison

	DH Divide-and-conquer	DH Semi-naive	MW
Pixelisation scheme	equiangular	equiangular	equiangular
Asymptotic complexity	$\mathcal{O}(L^{5/2} \log_2^{1/2} L)$	$\mathcal{O}(L^3)$	$\mathcal{O}(L^3)$
Precomputation	Y	N	N
Stability	N	Y	Y
Flexibility of Wigner recursion	N	N	Y
Spin functions	N	N	Y
Number of samples	$\mathbf{4L^2}$	$\mathbf{4L^2}$	$\mathbf{2L^2}$

Code to compute spherical harmonic transforms



SSHT code: Spin spherical harmonic transforms

`http://www.spinsht.org`

A novel sampling theorem on the sphere
McEwen & Wiaux (2011)

- Fortran, C, Matlab
- Supports scalar and spin functions on the sphere

Summary

- Observations on spherical manifolds are prevalent.
- Scale-discretized wavelets on the sphere afford the analysis of spatially localised, scale-dependent content and the exact synthesis of a function from its wavelet coefficients.
- Fast algorithms essential for the analysis of big data-sets.
- All codes publicly available (see `http://www.jasonmcewen.org`).
- Future work: by exploiting new sampling theorem on the sphere, we will develop yet more efficient algorithms.