Spin scale-discretised wavelets on the sphere for the analysis of CMB polarisation

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CMB polarisation

- CMB polarisation contains a wealth of cosmological information; but extracting this information is challenging.
- Evidence for primordial gravitational waves from BICEP2?
- Observe Q and U Stokes parameters.
- Construct $Q \pm \mathrm{i}U$, which is a spin ± 2 field: $(Q \pm \mathrm{i}U)'(\omega) = \mathrm{e}^{\mp \mathrm{i}2\chi}(Q \pm \mathrm{i}U)(\omega)$.
- Extract and analyse cosmological **maps**.

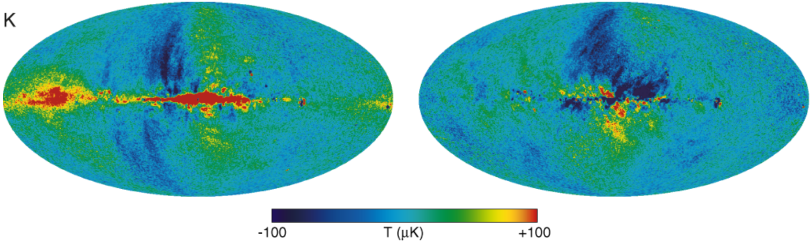

Figure: WMAP K-band Q and U maps.

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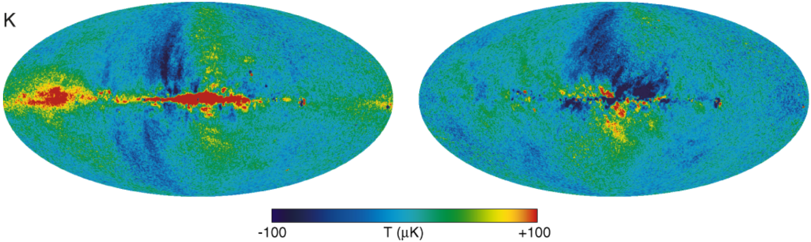

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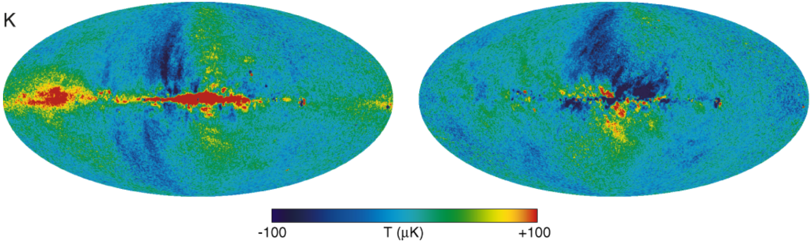

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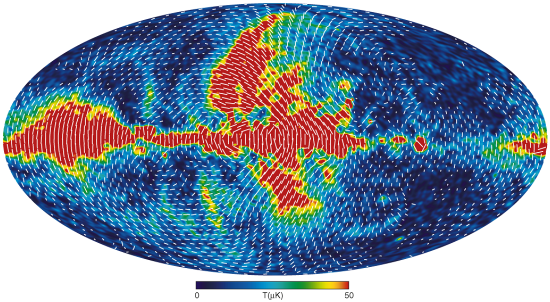

Figure: WMAP K-band $Q + \mathrm{i}U$ map.

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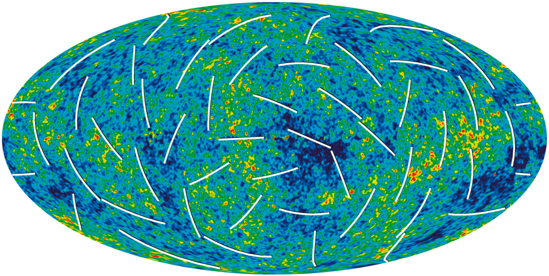

Figure: WMAP cosmological temperature and polarisation.

Outline

1. Spin scale-siscretised wavelets on the sphere

2. Fast algorithms

3. E/B separation

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Recall wavelet transform in Euclidean space

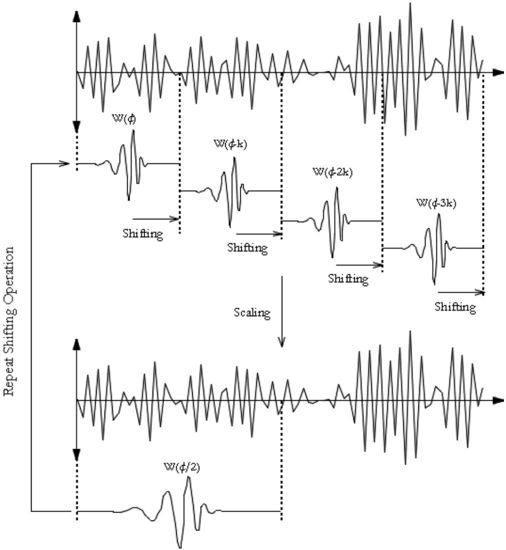

Figure: Wavelet scaling and shifting [Credit: `http://www.wavelet.org/tutorial/`]

Wavelets on the sphere

Dilation and translation

- Construct wavelet atoms from affine transformations (dilation, translation) on the sphere of a mother wavelet.
- The natural extension of translations to the sphere are rotations. Rotation of a function f on the sphere is defined by

$$\boxed{[\mathcal{R}(\rho)f](\omega) = f(\mathsf{R}_\rho^{-1} \cdot \omega), \quad \omega = (\theta, \varphi) \in \mathbb{S}^2, \quad \rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3)\ .}$$

translation

- How define dilation on the sphere?
    - Stereographic projection
      Antoine & Vandergheynst (1999), Wiaux *et al.* (2005)
    - Harmonic dilation wavelets
      McEwen *et al.* (2006), Sanz *et al.* (2006)
    - Isotropic undecimated wavelets
      Starck *et al.* (2005), Starck *et al.* (2009)
    - Needlets
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Scale-discretised wavelets on the sphere

Wavelet construction

- *Exact reconstruction with directional wavelets on the sphere*
  Wiaux, McEwen, Vandergheynst, Blanc (2008)

- Extend to spin functions.

- Scale-discretised wavelet ${}_s\Psi^j \in \mathrm{L}^2(\mathbb{S}^2)$ defined in harmonic space:

$$ {}_s\Psi^j_{\ell m} \equiv \kappa^j(\ell) s_{\ell m} \, . $$

- Admissible wavelets constructed to satisfy a resolution of the identity:

$$ \underbrace{\boxed{|{}_s\Phi_{\ell 0}|^2}}_{\text{scaling function}} + \sum_{j=0}^{J} \sum_{m=-\ell}^{\ell} \underbrace{\boxed{|{}_s\Psi^j_{\ell m}|^2}}_{\text{wavelet}} = 1 \, , \quad \forall \ell . $$

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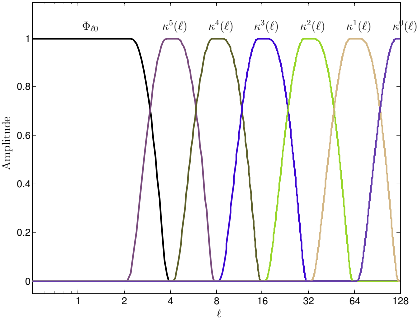

Figure: Harmonic tiling on the sphere.

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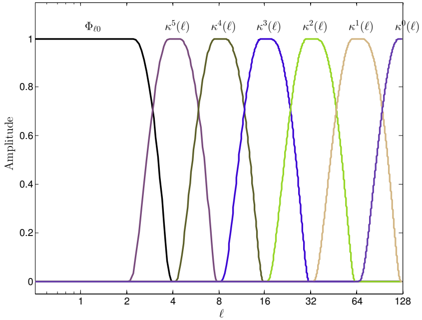

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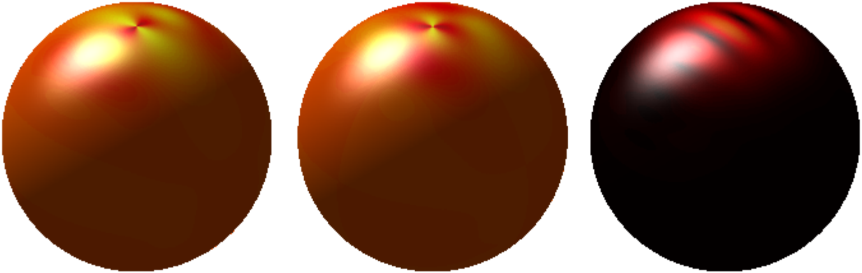

(a) Real$({}_s\Psi^j)$ (b) Imag$({}_s\Psi^j)$ (c) Abs$({}_s\Psi^j)$

Figure: Spin scale-discretised wavelets on the sphere.

Scale-discretised wavelets on the sphere

Forward and inverse transform (i.e. analysis and synthesis)

- The spin scale-discretised wavelet transform is given by the usual projection onto each wavelet:

$$\underbrace{\boxed{W^{{}_s\Psi^j}(\rho) = \langle {}_sf,\, \mathcal{R}_\rho\, {}_s\Psi^j \rangle}}_{\text{projection}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)\, {}_sf(\omega) (\mathcal{R}_\rho\, {}_s\Psi^j)^*(\omega) \; .$$

- Wavelet coefficients are scalar and not spin.

- Wavelet coefficients live in $\mathrm{SO}(3) \times \mathbb{Z}$; thus, directional structure is naturally incorporated.

- The original function may be recovered exactly in practice from the wavelet (and scaling) coefficients:

$${}_sf(\omega) = \underbrace{\boxed{\sum_{j=0}^{J}}}_{\text{finite sum}} \underbrace{\boxed{\int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{{}_s\Psi^j}(\rho) (\mathcal{R}_\rho\, {}_s\Psi^j)(\omega) \; .}}_{\text{wavelet contribution}}$$

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Steerability

- By imposing an azimuthal band-limit N, we recover steerable wavelets.
- By the linearity of the wavelet transform, steerability extends to wavelet coefficients:

$$W^{{}_s\Psi^j}(\alpha,\beta,\gamma) = \sum_{g=0}^{M-1} z(\gamma-\gamma_g)\, W^{{}_s\Psi^j}(\alpha,\beta,\gamma_g)\,.$$

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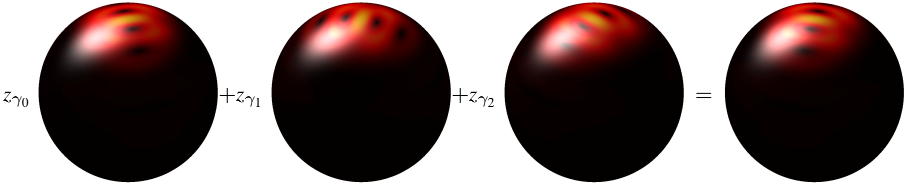

Figure: Steered wavelet computed from basis wavelets.

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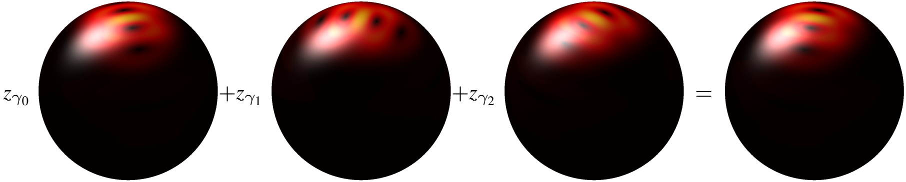

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1. Spin scale-siscretised wavelets on the sphere
2. Fast algorithms
3. E/B separation

Fast Wigner transform

Fourier transform on the rotation group $\mathrm{SO}(3)$

- Wavelet coefficients for scale j live on the rotation group $\mathrm{SO}(3)$: $W^{{}_s\Psi^j} \in \mathrm{L}^2(\mathrm{SO}(3))$.
- Develop fast wavelet transforms by considering their (Wigner) harmonic representation.

- Signal on the rotation group $F \in \mathrm{L}^2(\mathrm{SO}(3))$ may expressed by Wigner decomposition:

$$\boxed{F(\rho) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \frac{2\ell+1}{8\pi^2} F^{\ell}_{mn} D^{\ell *}_{mn}(\rho)}$$

where Wigner coefficients given by usual projection onto basis functions:

$$\boxed{F^{\ell}_{mn} = \langle F, D^{\ell *}_{mn} \rangle} = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) F(\rho)\, D^{\ell}_{mn}(\rho) \,.$$

- Novel sampling theorem on the rotation group requiring $2L^3$ samples only to represent a signal band-limited at L (follows straightforwardly from McEwen & Wiaux 2012).

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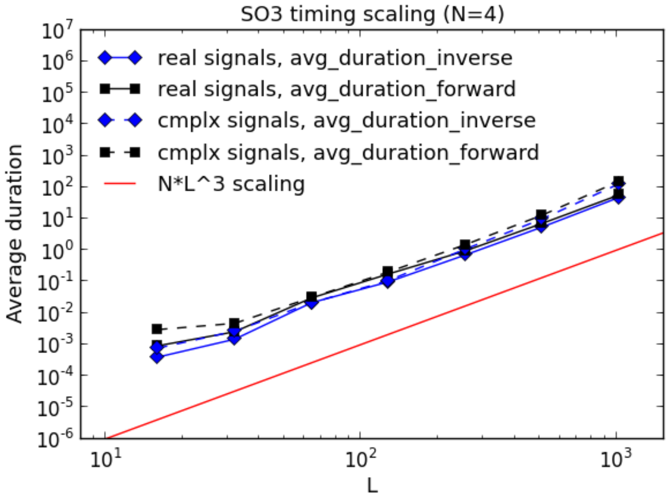

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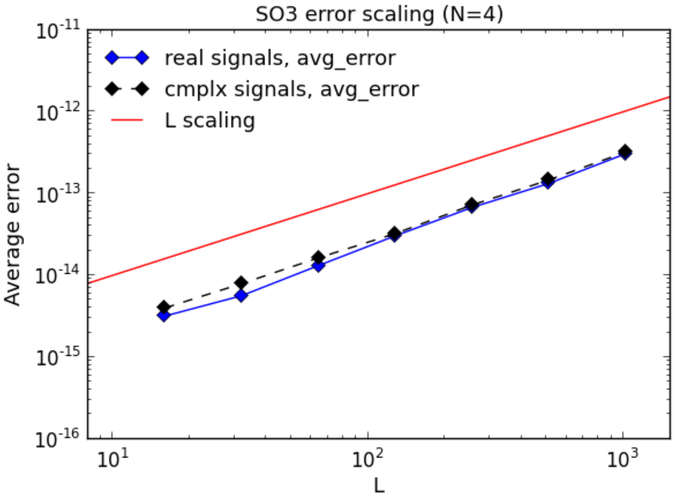

Fast spin scale-discretised wavelet transform

Exact and efficient computation via Wigner transforms

- Wavelet analysis can be posed as an inverse Wigner transform on $\mathrm{SO}(3)$:

$$\left(W^{{}_s\Psi^j}\right)^{\ell}_{mn} = \frac{8\pi^2}{2\ell+1}\, {}_sf_{\ell m}\, {}_s\Psi^{j*}_{\ell n}\,,$$

analysis

with

$$W^{{}_s\Psi^j}(\rho) = \sum_{\ell mn} \frac{2\ell+1}{8\pi^2} \left(W^{{}_s\Psi^j}\right)^{\ell}_{mn} D^{\ell *}_{mn}(\rho)\,.$$

- Wavelet synthesis can be posed as a forward Wigner transform on $\mathrm{SO}(3)$:

$${}_sf(\omega) = \sum_{j=0}^{J} \sum_{\ell mn} \frac{2\ell+1}{8\pi^2} \left(W^{{}_s\Psi^j}\right)^{\ell}_{mn}\, {}_s\Psi^{j}_{\ell n}\, {}_sY_{\ell m}(\omega)\,,$$

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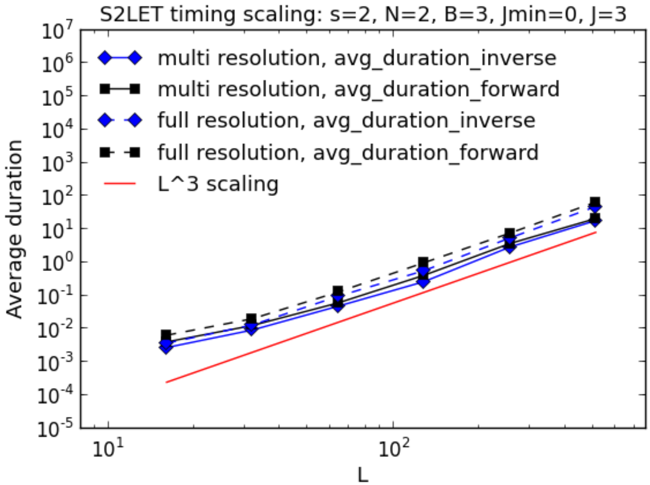

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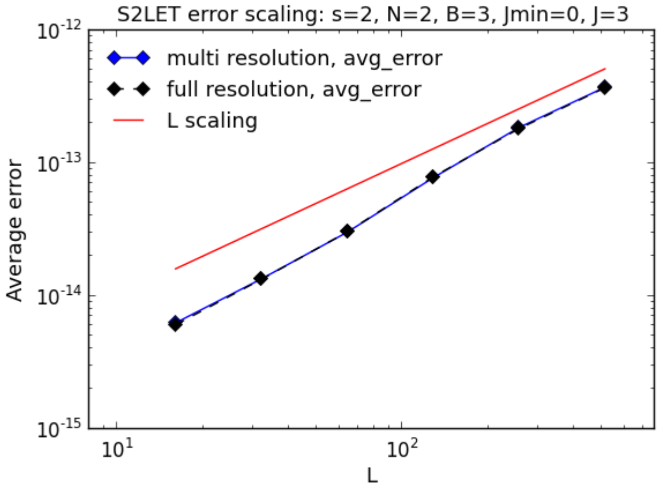

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E/B separation

From $Q \pm \mathrm{i}U$ to E and B maps

- Different physical processes exhibit different symmetries and thus behave differently under parity transformation.
- Can exploit this property to separate signals arising from different underlying physical mechanisms.
- Decompose $Q \pm \mathrm{i}U$ into parity even and odd components:

$$\boxed{\tilde{E}(\omega) = -\frac{1}{2}\left[\bar{\eth}^2(Q+\mathrm{i}U)(\omega) + \eth^2(Q-\mathrm{i}U)(\omega)\right]} \quad \text{E-mode}$$

$$\boxed{\tilde{B}(\omega) = \frac{\mathrm{i}}{2}\left[\bar{\eth}^2(Q+\mathrm{i}U)(\omega) - \eth^2(Q-\mathrm{i}U)(\omega)\right]} \quad \text{B-mode}$$

  where $\bar{\eth}$ and $\eth$ are spin lowering and raising operators, respectively.

- Number of existing techniques:
  Lewis *et al.* (2002), Bunn *et al.* (2003), Bowyer *et al.* (2011), Kim (2013).

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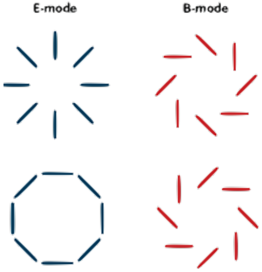


Figure: E-mode (even parity) and B-mode (odd parity) signals [Credit: `http://www.skyandtelescope.com/`].

- Number of existing techniques:
  Lewis *et al.* (2002), Bunn *et al.* (2003), Bowyer *et al.* (2011), Kim (2013).

E/B separation

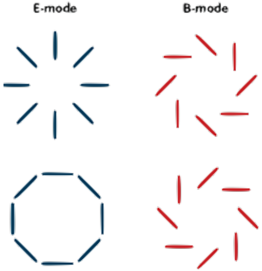
From $Q \pm \mathrm{i}U$ to E and B maps

- Different physical processes exhibit different symmetries and thus behave differently under parity transformation.
- Can exploit this property to separate signals arising from different underlying physical mechanisms.
- Decompose $Q \pm \mathrm{i}U$ into parity even and odd components:

$$\boxed{\tilde{E}(\omega) = -\frac{1}{2}\left[\bar{\eth}^2(Q+\mathrm{i}U)(\omega) + \eth^2(Q-\mathrm{i}U)(\omega)\right]} \quad \text{E-mode}$$

$$\boxed{\tilde{B}(\omega) = \frac{\mathrm{i}}{2}\left[\bar{\eth}^2(Q+\mathrm{i}U)(\omega) - \eth^2(Q-\mathrm{i}U)(\omega)\right]} \quad \text{B-mode}$$

where $\bar{\eth}$ and $\eth$ are spin lowering and raising operators, respectively.



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E/B separation

Spin and scalar scale-discretised wavelets

- Spin wavelet transform of $Q + \mathrm{i}U$ (observable):

$$\underbrace{\boxed{W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho) = \langle Q + \mathrm{i}U,\, \mathcal{R}_\rho \, {}_2\Psi^j \rangle}}_{\text{spin wavelet transform}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) (Q + \mathrm{i}U)(\omega) (\mathcal{R}_\rho \, {}_2\Psi^j)^*(\omega) \, .$$

- Scalar wavelet transforms of E and B (non-observable):

$$\underbrace{\boxed{W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = \langle \tilde{E},\, \mathcal{R}_\rho \, {}_0\tilde{\Psi}^j \rangle}}_{\text{scalar wavelet transform}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \tilde{E}(\omega) (\mathcal{R}_\rho \, {}_0\tilde{\Psi}^j)^*(\omega) \, ,$$

$$\underbrace{\boxed{W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = \langle \tilde{B},\, \mathcal{R}_\rho \, {}_0\tilde{\Psi}^j \rangle}}_{\text{scalar wavelet transform}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \tilde{B}(\omega) (\mathcal{R}_\rho \, {}_0\tilde{\Psi}^j)^*(\omega) \, ,$$

  where ${}_0\tilde{\Psi}^j \equiv \bar{\eth}^2 {}_2\Psi^j$.

- Spin wavelet coefficients of $Q + \mathrm{i}U$ are connected to scalar wavelet coefficients of E/B:

$$\boxed{W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = -\mathrm{Re}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \quad \text{and} \quad W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = -\mathrm{Im}\left[W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \, .}$$

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E/B separation

Using scale-discretised wavelets

Algorithm to recover E/B signals using scale-discretised wavelets

1. Compute spin wavelet transform of $Q + \mathrm{i}U$:

$$(Q+\mathrm{i}U)(\omega) \xrightarrow[\text{S2LET}]{\text{Spin wavelet transform}} W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)$$

2. Account for mask in harmonic and spatial domains simultaneously:

$$W^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho) \xrightarrow{\text{Mitigate mask}} \widehat{W}^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)$$

3. Construct E/B maps:

(a) $$W^{{}_0\tilde{\Psi}^j}_{\tilde{E}}(\rho) = -\mathrm{Re}\left[\widehat{W}^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \xrightarrow[\text{S2LET}]{\text{Inverse scalar wavelet transform}} \tilde{E}(\omega)$$

(b) $$W^{{}_0\tilde{\Psi}^j}_{\tilde{B}}(\rho) = -\mathrm{Im}\left[\widehat{W}^{{}_2\Psi^j}_{Q+\mathrm{i}U}(\rho)\right] \xrightarrow[\text{S2LET}]{\text{Inverse scalar wavelet transform}} \tilde{B}(\omega)$$

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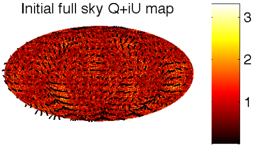
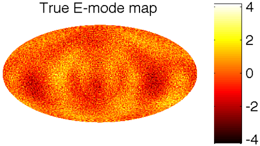
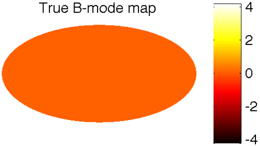
E/B separation
Simulations







E/B separation
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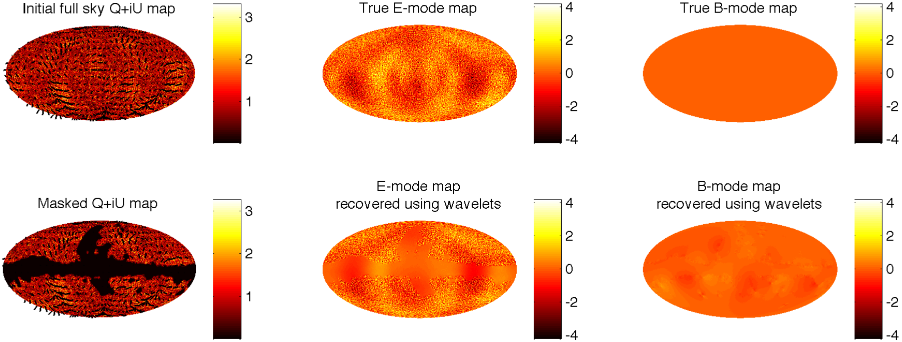
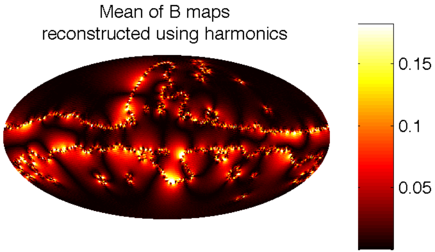
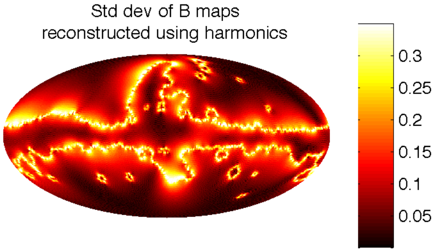


E/B separation

Preliminary results





E/B separation

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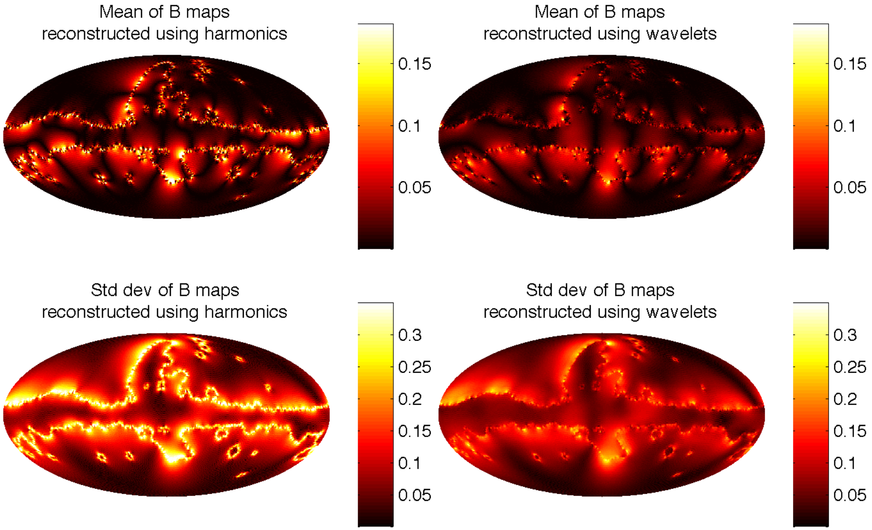
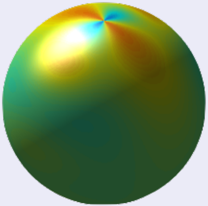


Summary

Spin scale-discretised wavelets are a powerful tool to study CMB polarisation, with an elegant and practical connection to scalar wavelet transforms of E/B maps.

S2LET code http://www.s2let.org



S2LET: A code to perform fast wavelet analysis on the sphere
Leistedt, McEwen, Vandergheynst, Wiaux (2012)

- C, Matlab, IDL, Java
- Supports only axisymmetric wavelets at present
- Extensions for directional and steerable wavelets, faster algos, and spin wavelets coming very soon

Extra Slides

Fast Wigner transform

Timing and accuracy