

Spin scale-discretised wavelets on the sphere for the analysis of CMB polarisation

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7th International Conference of the ERCIM on Computational and Methodological Statistics
Pisa, December 2014

CMB polarisation

- CMB polarisation contains a **wealth of cosmological information**; but extracting this information is **challenging**.
- Evidence for primordial gravitational waves from BICEP2?
- Observe Q and U Stokes parameters.
- Construct $Q \pm iU$, which is a **spin ± 2 field**: $(Q \pm iU)'(\omega) = e^{\mp i 2\chi}(Q \pm iU)(\omega)$.
- **Extract and analyse cosmological maps**.

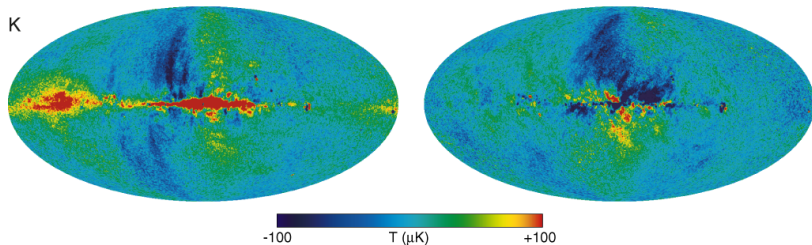


Figure: WMAP K-band Q and U maps.



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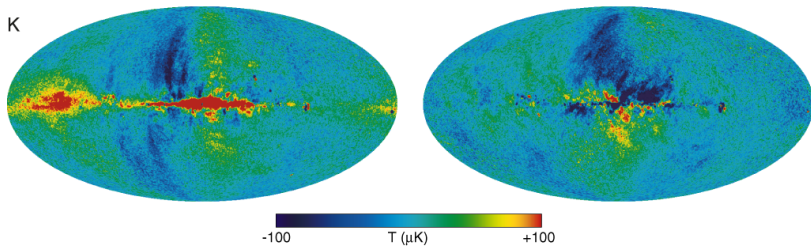


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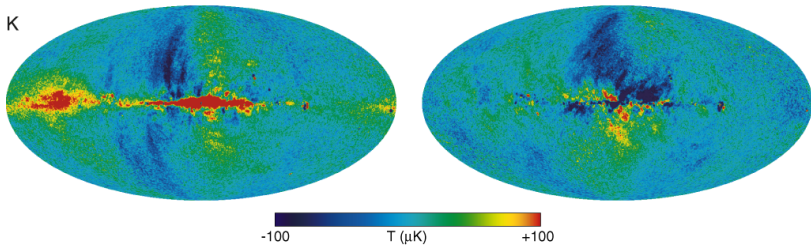


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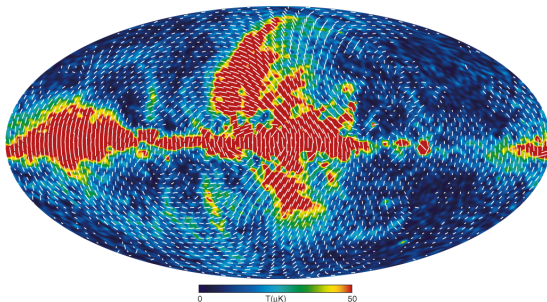


Figure: WMAP K-band $Q + iU$ map.



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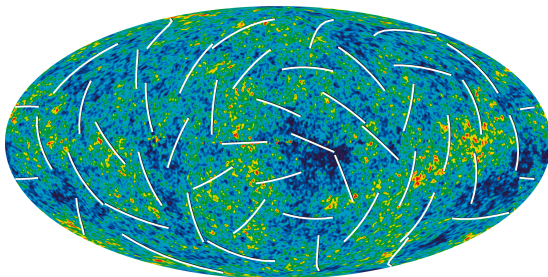


Figure: WMAP cosmological temperature and polarisation.



Outline

- 1 Spin scale-discretised wavelets on the sphere
- 2 Fast algorithms
- 3 E/B separation



Recall wavelet transform in Euclidean space

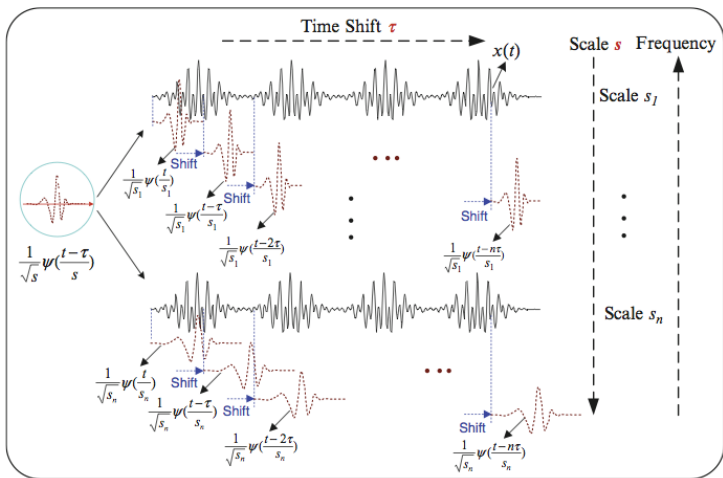


Figure: Wavelet scaling and shifting [Credit: Gao & Yan (2010)]



Wavelets on the sphere

Dilation and translation

- Construct **wavelet atoms from affine transformations** (dilation, translation) on the sphere of a mother wavelet.
- The natural **extension of translations to the sphere are rotations**. Rotation of a function f on the sphere is defined by

$$[\mathcal{R}(\rho)f](\omega) = f(\mathbf{R}_\rho^{-1} \cdot \omega), \quad \omega = (\theta, \varphi) \in \mathbb{S}^2, \quad \rho = (\alpha, \beta, \gamma) \in \text{SO}(3).$$

translation

- **How define dilation on the sphere?**
 - Stereographic projection
Antoine & Vandergheynst (1999), Wiaux *et al.* (2005)
 - Harmonic dilation wavelets
McEwen *et al.* (2006), Sanz *et al.* (2006)
 - Isotropic undecimated wavelets
Starck *et al.* (2005), Starck *et al.* (2009)
 - Needlets
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Scale-discretised wavelets on the sphere

Wavelet construction

- *Exact reconstruction with directional wavelets on the sphere*

Wiaux, McEwen, Vandergheynst, Blanc (2008)

- Extend to **spin functions**.

- Scale-discretised wavelet ${}_s\Psi^j \in L^2(S^2)$ defined in harmonic space:

$${}_s\Psi_{\ell m}^j \equiv \kappa^j(\ell) s_{\ell m}.$$

- Admissible wavelets constructed to satisfy a resolution of the identity:

$$\boxed{|{}_s\Phi_{\ell 0}|^2}_{\text{scaling function}} + \sum_{j=0}^J \sum_{m=-\ell}^{\ell} \boxed{|{}_s\Psi_{\ell m}^j|^2}_{\text{wavelet}} = 1, \quad \forall \ell.$$



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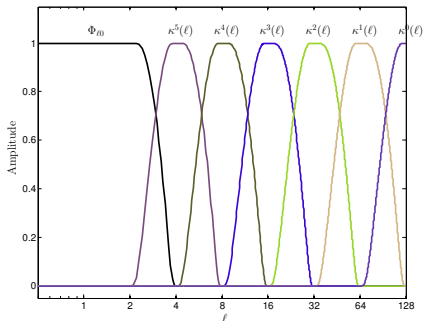


Figure: Harmonic tiling on the sphere.

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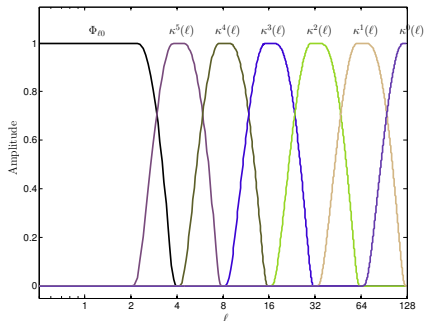


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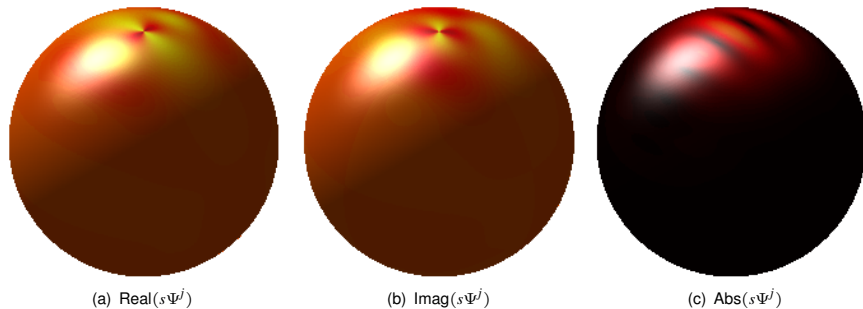


Figure: Spin scale-discretised wavelets on the sphere.



Scale-discretised wavelets on the sphere

Forward and inverse transform (i.e. analysis and synthesis)

- The **spin scale-discretised wavelet transform** is given by the usual projection onto each wavelet:

$$W^s \Psi^j(\rho) = \underbrace{\langle {}_s f, \mathcal{R}_{\rho} {}_s \Psi^j \rangle}_{\text{projection}} = \int_{\mathbb{S}^2} d\Omega(\omega) {}_s f(\omega) (\mathcal{R}_{\rho} {}_s \Psi^j)^*(\omega).$$

- Wavelet coefficients are scalar and not spin.
- Wavelet coefficients live in $SO(3) \times \mathbb{Z}$; thus, **directional structure is naturally incorporated**.
- The **original function may be recovered exactly in practice** from the wavelet (and scaling) coefficients:

$${}_s f(\omega) = \underbrace{\sum_{j=0}^J}_{\text{finite sum}} \underbrace{\int_{SO(3)} d\varrho(\rho) W^s \Psi^j(\rho) (\mathcal{R}_{\rho} {}_s \Psi^j)(\omega)}_{\text{wavelet contribution}}.$$



Fast Wigner transform

Fourier transform on the rotation group $SO(3)$

- Wavelet coefficients live on the rotation group $SO(3)$: $W_s \Psi^j \in L^2(SO(3))$.
- Develop fast wavelet transforms by considering their (Wigner) harmonic representation.
- Signal on the rotation group $F \in L^2(SO(3))$ may expressed by **Wigner decomposition**:

$$F(\rho) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \frac{2\ell+1}{8\pi^2} F_{mn}^{\ell} D_{mn}^{\ell*}(\rho)$$

where **Wigner coefficients** given by usual projection onto basis functions:

$$F_{mn}^{\ell} = \langle F, D_{mn}^{\ell*} \rangle = \int_{SO(3)} d\varrho(\rho) F(\rho) D_{mn}^{\ell}(\rho).$$

- **Novel sampling theorem on the rotation group** requiring $2L^3$ samples only to represent a signal band-limited at L (follows straightforwardly from McEwen & Wiaux 2012).



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Fast spin scale-discretised wavelet transform

Exact and efficient computation via Wigner transforms

- Wavelet analysis can be posed as an inverse Wigner transform on $SO(3)$:

$$(W^{\Psi^j})_{mn}^{\ell} = \frac{8\pi^2}{2\ell + 1} f_{\ell m} \Psi_{\ell n}^{j*} .$$

analysis

- Wavelet synthesis can be posed as a forward Wigner transform on $SO(3)$:

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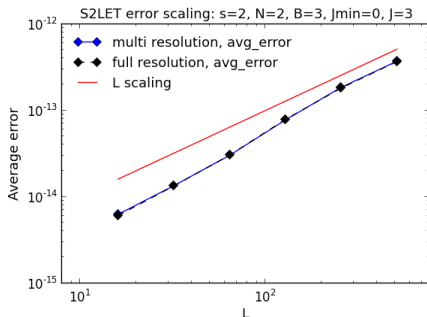
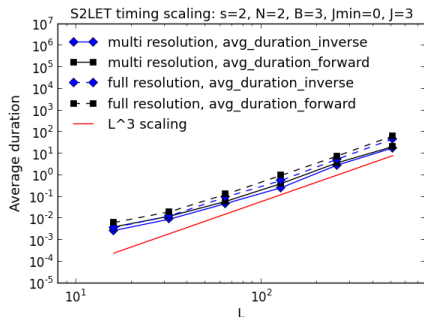
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Fast spin scale-discretised wavelet transform

Timing and accuracy



E/B separation

From $Q \pm iU$ to E and B maps

- Different physical processes exhibit different symmetries and thus behave differently under parity transformation.
- Can exploit this property to separate signals arising from different underlying physical mechanisms.
- Decompose $Q \pm iU$ into parity even and odd components:

$$\tilde{E}(\omega) = -\frac{1}{2} \left[\bar{\partial}^2(Q + iU)(\omega) + \partial^2(Q - iU)(\omega) \right]$$

E-mode

$$\tilde{B}(\omega) = \frac{i}{2} \left[\bar{\partial}^2(Q + iU)(\omega) - \partial^2(Q - iU)(\omega) \right]$$

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where $\bar{\partial}$ and ∂ are spin lowering and raising operators, respectively.

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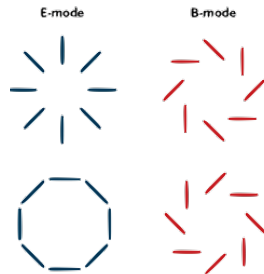


Figure: E-mode (even parity) and B-mode (odd parity) signals [Credit: <http://www.skyandtelescope.com/>].



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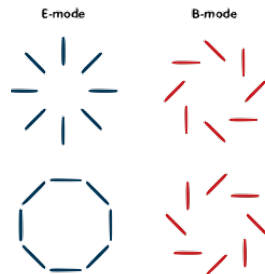


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E/B separation

Using scale-discretised wavelets

Algorithm to recover E/B signals using scale-discretised wavelets

- 1 Compute spin wavelet transform of $Q + iU$:

$$(Q + iU)(\omega) \xrightarrow[\text{S2LET}]{\text{Spin wavelet transform}} W_{Q+iU}^{2\Psi^j}(\rho)$$

- 2 Account for mask in **harmonic and spatial** domains simultaneously:

$$W_{Q+iU}^{2\Psi^j}(\rho) \xrightarrow{\text{Mitigate mask}} \widehat{W}_{Q+iU}^{2\Psi^j}(\rho)$$

- 3 Construct E/B maps:

$$(a) \quad W_{\tilde{E}}^{0\tilde{\Psi}^j}(\rho) = -\text{Re} \left[\widehat{W}_{Q+iU}^{2\Psi^j}(\rho) \right] \xrightarrow[\text{S2LET}]{\text{Inverse scalar wavelet transform}} \tilde{E}(\omega)$$

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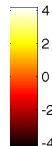
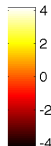
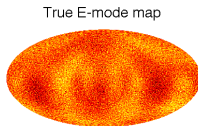
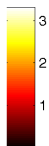
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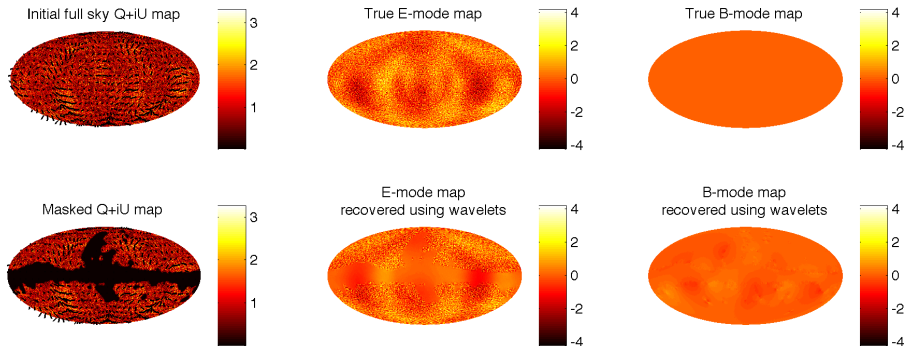
E/B separation

Simulations



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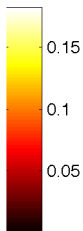
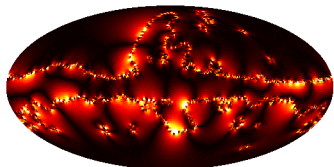
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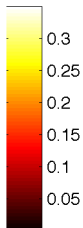
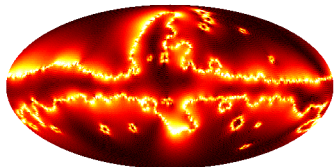
E/B separation

Preliminary results

Mean of B maps
reconstructed using harmonics



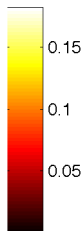
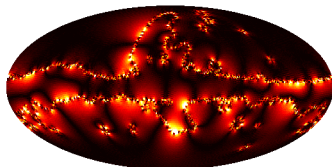
Std dev of B maps
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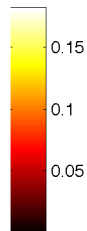
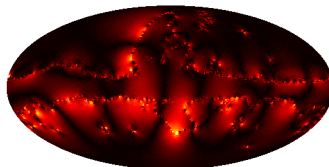
E/B separation

Preliminary results

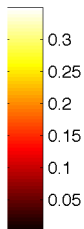
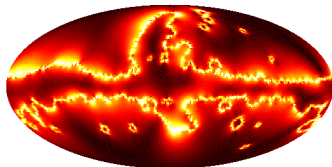
Mean of B maps
reconstructed using harmonics



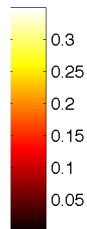
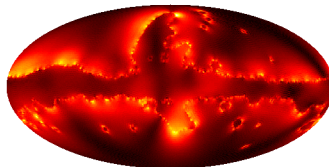
Mean of B maps
reconstructed using wavelets



Std dev of B maps
reconstructed using harmonics



Std dev of B maps
reconstructed using wavelets

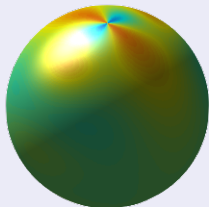


Summary

Spin scale-discretised wavelets are a powerful tool to study CMB polarisation, with an *elegant* and *practical* connection to scalar wavelet transforms of E/B maps.

S2LET code

<http://www.s2let.org>



S2LET: A code to perform fast wavelet analysis on the sphere

Leistedt, McEwen, Vanderghaynst, Wiaux (2012)

- C, Matlab, IDL, Java
- Supports only axisymmetric wavelets at present
- Extensions for directional and steerable wavelets, faster algos, and spin wavelets coming very soon

