Implications of a new sampling theorem for sparse signal reconstruction on the sphere

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arXiv:1110.6298

arXiv:1205.1013

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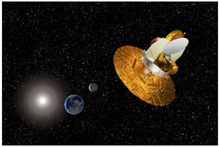
Observations of the cosmic microwave background (CMB)

- Full-sky observations of the CMB ongoing.

(a) COBE (launched 1989)



(b) WMAP (launched 2001)

(c) Planck (launched 2009)

- Each new experiment provides dramatic improvement in precision and resolution of observations.

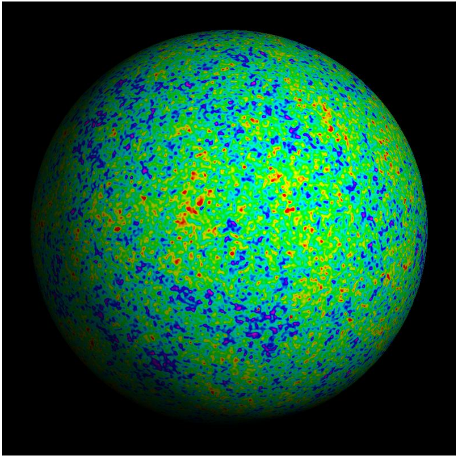
(cobe 2 wmap movie)

(d) COBE to WMAP [Credit: WMAP Science Team]

(planck movie)

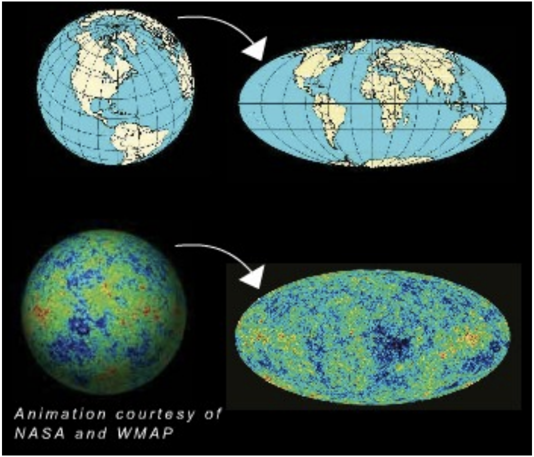
(e) Planck observing strategy [Credit: Planck Collaboration]

Observations of the cosmic microwave background (CMB)



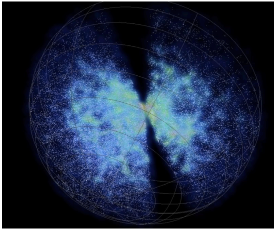
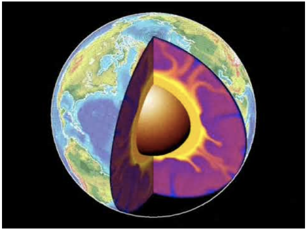
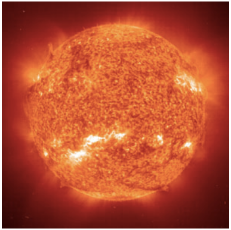
Credit: Max Tegmark

Observations on the sphere



Credit: Alec MacAndrew

Observations on the three-ball (solid sphere)

- Boris Leistedt & JDM (2012), *Exact wavelets on the ball*, submitted to IEEE Trans. Sig. Proc., arXiv:1205.0792.

Outline

1. Harmonic analysis on the sphere
   - Spherical harmonic transform
2. Sampling theorems on the sphere
   - Driscoll & Healy sampling theorem
   - McEwen & Wiaux sampling theorem
   - Comparison
3. Sparse signal reconstruction on the sphere
   - Sparse signal reconstruction
   - TV inpainting
   - Low-resolution simulations
   - High-resolution simulations
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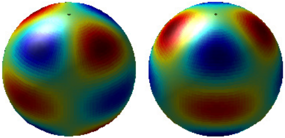
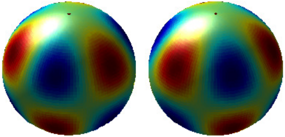
Spherical harmonic transform

- The spherical harmonics are the eigenfunctions of the Laplacian on the sphere: $\Delta_{\mathrm{S}^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$.



(i) $\ell = 4, m = 2$



(j) $\ell = 4, m = 3$

Figure: Spherical harmonic functions (real and imaginary parts).

- Any square integrable scalar function on the sphere $f \in \mathrm{L}^2(\mathrm{S}^2)$ may be represented by its spherical harmonic expansion:

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} \, Y_{\ell m}(\theta, \varphi) \, .$$

- The spherical harmonic coefficients are given by the usual projection onto each basis function:

$$f_{\ell m} = \langle f, \, Y_{\ell m} \rangle = \int_{\mathrm{S}^2} \mathrm{d}\Omega(\theta, \varphi) \, f(\theta, \varphi) \, Y_{\ell m}^*(\theta, \varphi) \, .$$

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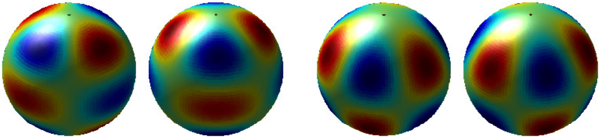

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Spherical harmonic transform

- We consider signals on the sphere band-limited at L, that is signals such that $f_{\ell m} = 0, \forall \ell \geq L$
  $\Rightarrow$ summations may be truncated at $L-1$:

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- For a band-limited signal, can we compute $f_{\ell m}$ exactly?
  $\rightarrow$ Sampling theorems on the sphere.

- Aside: Generalise to spin functions on the sphere.

  Square integrable spin functions on the sphere ${}_sf \in \mathrm{L}^2(\mathbb{S}^2)$, with integer spin $s \in \mathbb{Z}$, are defined by their behaviour under local rotations. By definition, a spin function transforms as

$${}_sf'(\theta, \varphi) = \mathrm{e}^{-\mathrm{i}s\chi} \; {}_sf(\theta, \varphi)$$

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Driscoll & Healy (DH) sampling theorem

- Canonical sampling theorem on the sphere derived by Driscoll & Healy (1994) for equiangular grids.
- Gives an explicit quadrature rule for the spherical harmonic transform:

$$f_{\ell m} = \sum_{t=0}^{2L-1} \sum_{p=0}^{2L-1} q_{\mathrm{DH}}(\theta_t)\, f(\theta_t, \varphi_p)\, Y_{\ell m}^*(\theta_t, \varphi_p) \;,$$

where the sample positions are defined by $\theta_t = \pi t / 2L$, for $t = 0, \ldots, 2L-1$, and $\varphi_p = \pi p / L$, for $p = 0, \ldots, 2L-1$

$\Rightarrow$ $N_{\mathrm{DH}} = (2L-1)2L + 1 \sim 4L^2$ samples on the sphere.

- The quadrature weights are defined implicitly by the solution to

$$\sum_{t=0}^{2L-1} q_{\mathrm{DH}}(\theta_t)\, P_\ell(\cos\theta_t) = \frac{2\pi}{L}\, \delta_{\ell 0} \;, \quad \forall \ell < 2L \;,$$

and are given explicitly by

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(a) Function on sphere (b) Even function on torus (c) Odd function on torus

Figure: Associating functions on the sphere and torus

McEwen & Wiaux (MW) sampling theorem

- By a factoring of rotations, a reordering of summations and a separation of variables, the inverse transform of ${}_sf$ may be written:

Inverse spherical harmonic transform

$${}_sf(\theta, \varphi) = \sum_{m=-(L-1)}^{L-1} {}_sF_m(\theta)\, e^{im\varphi}$$

$${}_sF_m(\theta) = \sum_{m'=-(L-1)}^{L-1} {}_sF_{mm'}\, e^{im'\theta}$$

$${}_sF_{mm'} = (-1)^s\, i^{-(m+s)} \sum_{\ell=0}^{L-1} \sqrt{\frac{2\ell+1}{4\pi}}\, \Delta^{\ell}_{m'm}\, \Delta^{\ell}_{m',-s}\, {}_sf_{\ell m}$$

where $\Delta^{\ell}_{mn} \equiv d^{\ell}_{mn}(\pi/2)$ are the reduced Wigner functions evaluated at $\pi/2$.

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- By a factoring of rotations, a reordering of summations and a separation of variables, the forward transform of ${}_sf$ may be written:

Forward spherical harmonic transform

$$ {}_sf_{\ell m} = (-1)^s \, \mathrm{i}^{m+s} \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'=-(L-1)}^{L-1} \Delta^{\ell}_{m'm} \, \Delta^{\ell}_{m',-s} \, {}_sG_{mm'} $$

$$ {}_sG_{mm'} = \int_0^{\pi} \mathrm{d}\theta \sin\theta \, {}_sG_m(\theta) \, \mathrm{e}^{-\mathrm{i}m'\theta} $$

$$ {}_sG_m(\theta) = \int_0^{2\pi} \mathrm{d}\varphi \, {}_sf(\theta,\varphi) \, \mathrm{e}^{-\mathrm{i}m\varphi} $$

- JDM (2011a), *Fast, exact (but unstable) spin spherical harmonic transforms*
- Huffenberger & Wandelt (2010), *Fast and exact spin-s spherical harmonic transforms*
- JDM & Wiaux (2011b), *A novel sampling theorem on the sphere*

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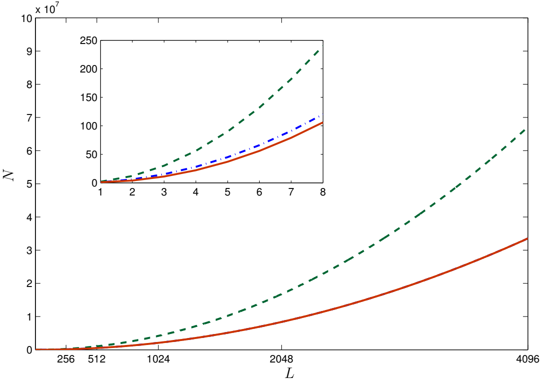

Figure: Number of samples (MW=red; DH=green; GL=blue)

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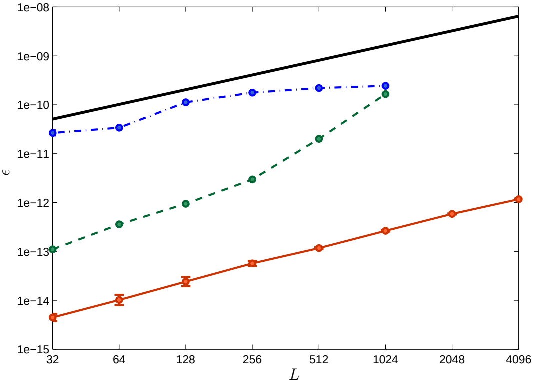

Figure: Numerical accuracy (MW=red; DH=green; GL=blue)

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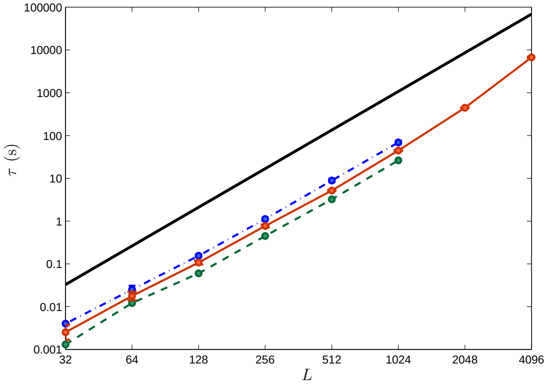

Figure: Computation time (MW=red; DH=green; GL=blue)

Comparison

	DH Divide-and-conquer	DH Semi-naive	MW
Pixelisation scheme	equiangular	equiangular	equiangular
Asymptotic complexity	$\mathcal{O}(L^{5/2} \log_2^{1/2} L)$	$\mathcal{O}(L^3)$	$\mathcal{O}(L^3)$
Precomputation	Y	N	N
Stability	N	Y	Y
Flexibility of Wigner recursion	N	N	Y
Spin functions	N	N	Y
Number of samples	$\mathbf{4L^2}$	$\mathbf{4L^2}$	$\mathbf{2L^2}$

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Sparse signal reconstruction on the sphere

- A reduction in the number of samples required to represent a band-limited signal on the sphere has important implications for sparse signal reconstruction.
- Many natural signals are sparse in a spatially localised measure, such as in a wavelet basis, overcomplete dictionary, or in the magnitude of their gradient, for example.
- A more efficient sampling of a band-limited signal on the sphere improves both the dimensionality and sparsity of the signal in the spatial domain.
- For a given number of measurements, a more efficient sampling theorem improves the fidelity of sparse signal reconstruction.
- We develop a framework for total variation (TV) inpainting on the sphere to demonstrate this result.

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TV inpainting

- Consider inpainting problem $\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n}$ in the context of different sampling theorems, where:
  - the samples of f are denoted by the concatenated vector $\boldsymbol{x} \in \mathbb{R}^N$;
  - N is the number of samples on the sphere of the chosen sampling theorem;
  - M noisy measurements $\boldsymbol{y} \in \mathbb{R}^M$ are acquired;
  - the measurement operator $\Phi \in \mathbb{R}^{M \times N}$ represents a random masking of the signal;
  - the noise $\boldsymbol{n} \in \mathbb{R}^M$ is assumed to be iid Gaussian with zero mean.
- Define TV norm on the sphere:

$$\int_{\mathrm{S}^2} \mathrm{d}\Omega \, |\nabla f| \simeq \sum_{t=0}^{N_\theta - 1} \sum_{p=0}^{N_\varphi - 1} |\nabla f| \, q(\theta_t) \simeq \sum_{t=0}^{N_\theta - 1} \sum_{p=0}^{N_\varphi - 1} \sqrt{q^2(\theta_t)(\delta_\theta \boldsymbol{x})^2 + \frac{q^2(\theta_t)}{\sin^2 \theta_t}(\delta_\varphi \boldsymbol{x})^2} \equiv \|\boldsymbol{x}\|_{\mathrm{TV}} .$$

- TV inpainting problem solved directly on the sphere:

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_{\mathrm{TV}} \text{ such that } \|\boldsymbol{y} - \Phi \boldsymbol{x}\|_2 \leq \epsilon .$$

- TV inpainting problem solved in harmonic space:

$$\hat{\boldsymbol{x}}'^\star = \arg\min_{\hat{\boldsymbol{x}}'} \|\Lambda' \hat{\boldsymbol{x}}'\|_{\mathrm{TV}} \text{ such that } \|\boldsymbol{y} - \Phi \Lambda' \hat{\boldsymbol{x}}'\|_2 \leq \epsilon ,$$

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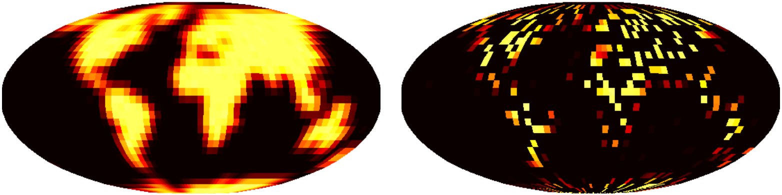
TV inpainting: low-resolution simulations

- Solve TV inpainting problem on the sphere in the context of the Driscoll & Healy sampling theorem and our new sampling theorem (at $L = 32$).

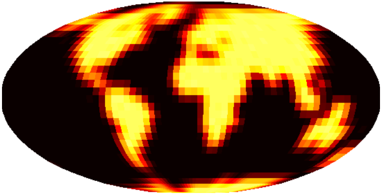


(a) Ground truth (b) Measurements

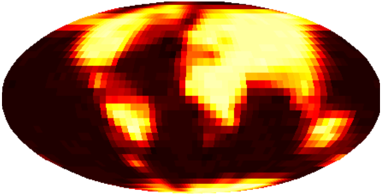
Figure: Earth topographic data reconstructed in the harmonic domain for $M/L^2 = 1/2$

TV inpainting: low-resolution simulations

- Solve TV inpainting problem on the sphere in the context of the Driscoll & Healy sampling theorem and our new sampling theorem (at $L = 32$).



(a) Ground truth



(b) DH reconstruction

Figure: Earth topographic data reconstructed in the harmonic domain for $M/L^2 = 1/2$

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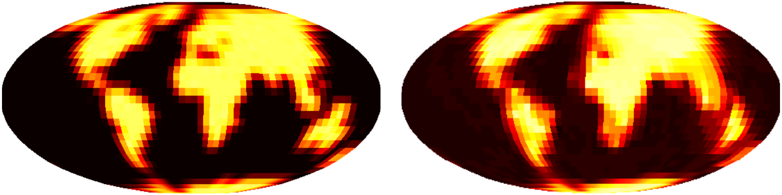
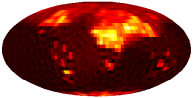


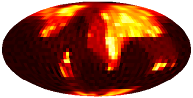
(a) Ground truth (b) MW reconstruction

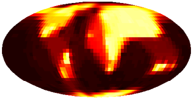
Figure: Earth topographic data reconstructed in the harmonic domain for $M/L^2 = 1/2$

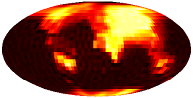
TV inpainting: low-resolution simulations

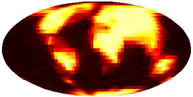

(a) DH spatial for $\frac{M}{L^2} = \frac{1}{4}$

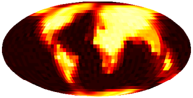

(b) DH harmonic for $\frac{M}{L^2} = \frac{1}{4}$

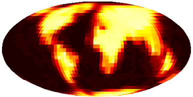

(c) MW spatial for $\frac{M}{L^2} = \frac{1}{4}$

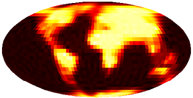

(d) MW harmonic for $\frac{M}{L^2} = \frac{1}{4}$

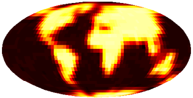

(e) DH spatial for $\frac{M}{L^2} = \frac{1}{2}$

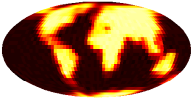

(f) DH harmonic for $\frac{M}{L^2} = \frac{1}{2}$

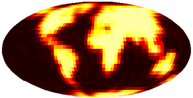

(g) MW spatial for $\frac{M}{L^2} = \frac{1}{2}$

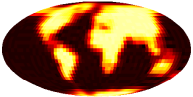

(h) MW harmonic for $\frac{M}{L^2} = \frac{1}{2}$

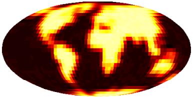

(i) DH spatial for $\frac{M}{L^2} = 1$

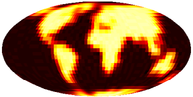

(j) DH harmonic for $\frac{M}{L^2} = 1$

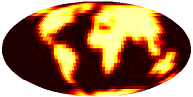

(k) MW spatial for $\frac{M}{L^2} = 1$


(l) MW harmonic for $\frac{M}{L^2} = 1$


(m) DH spatial for $\frac{M}{L^2} = \frac{3}{2}$


(n) DH harmonic for $\frac{M}{L^2} = \frac{3}{2}$


(o) MW spatial for $\frac{M}{L^2} = \frac{3}{2}$


(p) MW harmonic for $\frac{M}{L^2} = \frac{3}{2}$

TV inpainting: low-resolution simulations

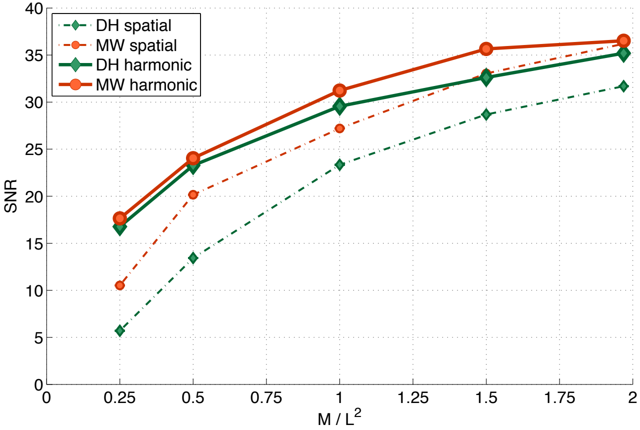


Figure: Reconstruction performance for the DH and MW sampling theorems

TV inpainting: high-resolution simulations

- Previously limited to low-resolution simulations.
- To solve high-resolution problem we require fast adjoint spherical harmonic transform operators in addition to fast forward spherical harmonic transforms to solve optimisation problems.
- Superiority of new sampling theorem clear, hence develop fast adjoints for this case only.

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Fast adjoint inverse spherical harmonic transform

$$ {}_s\tilde{f}^\dagger(\theta_t, \varphi_p) = \begin{cases} {}_sf(\theta_t, \varphi_p)\,, & t \in \{0, 1, \ldots, L-1\} \\ 0\,, & t \in \{L, \ldots, 2L-2\} \end{cases} $$

$$ {}_sF_{mm'}{}^\dagger = \sum_{t=0}^{2L-2} \sum_{p=0}^{2L-2} {}_s\tilde{f}^\dagger(\theta_t, \varphi_p)\, \mathrm{e}^{-\mathrm{i}(m'\theta_t + m\varphi_p)} $$

$$ {}_sf_{\ell m}{}^\dagger = (-1)^s\, \mathrm{i}^{m+s} \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'=-(L-1)}^{L-1} \Delta^\ell_{m'm}\, \Delta^\ell_{m',-s}\, {}_sF_{mm'}{}^\dagger $$

TV inpainting: high-resolution simulations

Fast adjoint forward spherical harmonic transform

$$ {}_sG_{mm'}{}^{\dagger} = (-1)^s \, \mathrm{i}^{-(m+s)} \sum_{\ell=0}^{L-1} \sqrt{\frac{2\ell+1}{4\pi}} \, \Delta^{\ell}_{m'm} \, \Delta^{\ell}_{m',-s} \, {}_sf_{\ell m} $$

$$ {}_sF_{mm''}{}^{\dagger} = 2\pi \sum_{m'=-(L-1)}^{L-1} {}_sG_{mm'}{}^{\dagger} \, w(m'-m'') $$

$$ {}_s\tilde{F}_m{}^{\dagger}(\theta_t) = \frac{1}{2L-1} \sum_{m'=-(L-1)}^{L-1} {}_sF_{mm'}{}^{\dagger} \, \mathrm{e}^{\mathrm{i}m'\theta_t} $$

$$ {}_sF_m{}^{\dagger}(\theta_t) = \begin{cases} {}_s\tilde{F}_m{}^{\dagger}(\theta_t) + (-1)^{m+s} \, {}_s\tilde{F}_m{}^{\dagger}(\theta_{2L-2-t}) \, , & t \in \{0, 1, \dots, L-2\} \\ {}_s\tilde{F}_m{}^{\dagger}(\theta_t) \, , & t = L-1 \end{cases} $$

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TV inpainting: high-resolution simulations

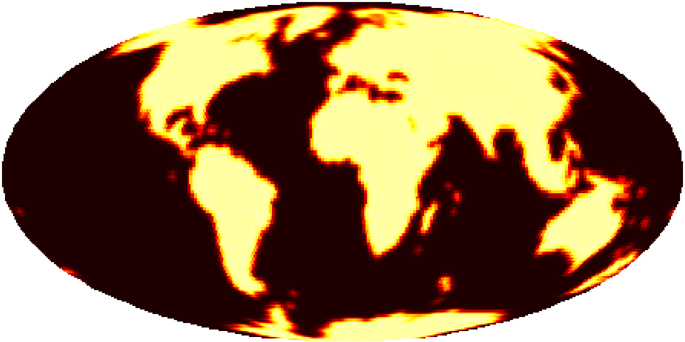

Figure: Ground truth ($L = 128$)

TV inpainting: high-resolution simulations

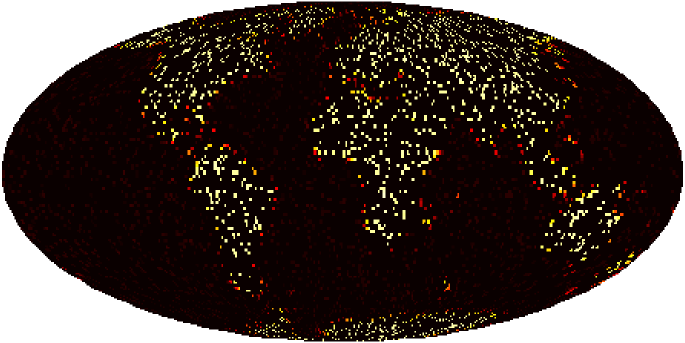

Figure: Measurements ($M/L^2 = 1/4; L = 128$)

TV inpainting: high-resolution simulations

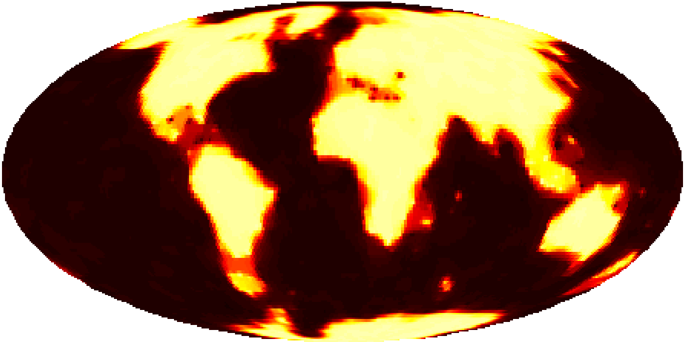

Figure: Reconstruction ($M/L^2 = 1/4$; $L = 128$; SNR $= 29$dB)

Outline

1. Harmonic analysis on the sphere
   - Spherical harmonic transform
2. Sampling theorems on the sphere
   - Driscoll & Healy sampling theorem
   - McEwen & Wiaux sampling theorem
   - Comparison
3. Sparse signal reconstruction on the sphere
   - Sparse signal reconstruction
   - TV inpainting
   - Low-resolution simulations
   - High-resolution simulations
4. Summary

Summary

- New MW sampling theorem on the sphere which captures all of the information content of a band-limited signal in only $2L^2$ samples (compared to $4L^2$ for the DH sampling theorem).
- A reduction in the number of samples required to represent a band-limited signal on the sphere has important implications for sparse signal reconstruction.
- For signals sparse in a spatially localised representation, a more efficient sampling of the sphere improves the fidelity of sparse signal reconstruction.
- We develop a framework for total variation (TV) inpainting on the sphere to demonstrate this result $\rightarrow$ superiority of the MW sampling theorem for sparse signal reconstruction clear.
- Develop fast adjoint spherical harmonic transforms for the MW sampling theorem to solve sparse signal reconstruction problems on the sphere at high-resolution.

Papers

- McEwen & Wiaux, *A novel sampling theorem on the sphere*, IEEE Trans. Sig. Proc., 59, 12, 5876–5887, arXiv:1110.6298, 2011.
- McEwen, Puy, Thiran, Vandergheynst, Van De Ville & Wiaux, *Sparse signal reconstruction on the sphere: implications of a new sampling theorem*, IEEE Trans. Sig. Proc., submitted, arXiv:1205.1013, 2012.

Code

SSHT Code to compute fast and exact, forward and adjoint (spin) spherical harmonic transforms based on the MW sampling theorem
(Fortran, C, Matlab)

Available under the GPL from `http://www.ssht.org.uk/`

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