Radio Interferometric Imaging with Uncertainties

Xiaohao Cai*, Marcelo Pereyra†, and Jason D. McEwen*
*Mullard Space Science Laboratory (MSSL), University College London, UK
†Maxwell Institute for Mathematical Sciences, Heriot-Watt University, UK

***Abstract*—Uncertainty quantification (UQ) is a critical missing component in radio interferometric (RI) imaging that will only become increasingly important as the big-data era of radio interferometry emerges. Statistical sampling approaches to perform Bayesian inference, like Markov Chain Monte Carlo (MCMC) sampling, can in principle recover the full posterior distribution of the image, from which uncertainties can then be quantified. However, for massive data sizes, it will be difficult if not impossible to apply any MCMC technique due to its inherent computational cost. We formulate Bayesian inference problems with sparsity-promoting priors (motivated by compressive sensing), for which we recover *maximum a posteriori* (MAP) point estimators of RI images by convex optimisation. Exploiting recent developments in the theory of probability concentration, uncertainties can also be quantified by post-processing the recovered MAP estimate. In this work, we review two UQ methodologies [1], [2] – respectively based on MCMC techniques and MAP estimation – with application in RI imaging.**

I. Introduction

Let $\boldsymbol{y} \in \mathbb{C}^M$ be the M visibilities acquired by a radio interferometric (RI) telescope observed under a linear measurement operator $\boldsymbol{\Phi} \in \mathbb{C}^{M\times N}$ modelling the acquisition of the sky brightness distribution $\boldsymbol{x}$ ($\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{a}$, where $\boldsymbol{\Psi} \in \mathbb{C}^{N\times L}$ is a (wavelet) basis and $\boldsymbol{a}$, a sparse vector, represents the associated coefficients of $\boldsymbol{x}$ under $\boldsymbol{\Psi}$). Then, we have

$$\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{n}, \tag{1}$$

where $\boldsymbol{n} \in \mathbb{C}^M$ is the instrumental noise. In practice, $\boldsymbol{y}$ is only observed partially or with limited resolution. Recovering the sky intensity signal $\boldsymbol{x}$ from $\boldsymbol{y}$ according to equation (1) then amounts to solving an ill-conditioned inverse problem [3].

The RI inverse problem (1) can be solved elegantly in the Bayesian framework [1]. After combining the observed and prior information using ℓ_2-norm (likelihood) and ℓ_1-norm (prior), the posterior distribution $p(\boldsymbol{x}|\boldsymbol{y})$ can be obtained by using Bayes' theorem, i.e.,

$$p(\boldsymbol{x}|\boldsymbol{y}) \propto \exp\Big\{-\big(\mu\|\boldsymbol{\Psi}^\dagger\boldsymbol{x}\|_1 + \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2/2\sigma^2\big)\Big\}. \tag{2}$$

In particular, it is often common practice to compute the maximum-a-posteriori (MAP) estimator given by

$$\boldsymbol{x}_{\rm map} = \underset{\boldsymbol{x}}{\operatorname{argmin}} \Big\{\mu\|\boldsymbol{\Psi}^\dagger\boldsymbol{x}\|_1 + \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2/2\sigma^2\Big\}. \tag{3}$$

II. Uncertainty qualification (UQ) methodologies

Let $C_\alpha \subset \mathbb{R}^N$ with $\alpha \in (0,1)$ be a posterior credible region – here we consider the highest posterior density (HPD) region [1] – with confidence level $100(1-\alpha)\%$, where $p(\boldsymbol{x} \in C_\alpha|\boldsymbol{y}) = 1-\alpha$. Two ways can be used to find/estimate the HPD region C_α: one is using the MCMC samples [1] obtained by sampling the posterior distribution in (2), the other is based on the MAP estimation [2] given in (3). Moreover, these two different ways motivated us to develop two UQ methodologies, shown in Figs. 1 and 2, respectively.

The light green areas in Figs. 1 and 2 on the right show the types of UQ developed. Firstly, full posterior samples (from which a point estimator say $\boldsymbol{x}^*$ can also be generated by e.g. using the sample mean) and an MAP estimator are obtained by using MCMC techniques and convex optimisation techniques, respectively. Then, various forms of UQ are performed. After HPD credible regions and its approximation (C'_α) are computed, they are then used to compute local credible intervals (*cf.* error bars), namely $(\boldsymbol{\xi}_-, \boldsymbol{\xi}_+)$, corresponding to individual pixels and superpixels, and to perform hypothesis testing of image structure to test whether a structure is physical or an artefact.

A brief summary in performance is: methodology I is able to provide benchmark results but is computationally expensive; methodology II is a trustworthy approximation and is approximately 10^5 times faster computationally than methodology I, and therefore will scale to big-data (like those anticipated from SKA); refer to [1], [2] and references therein for more details.

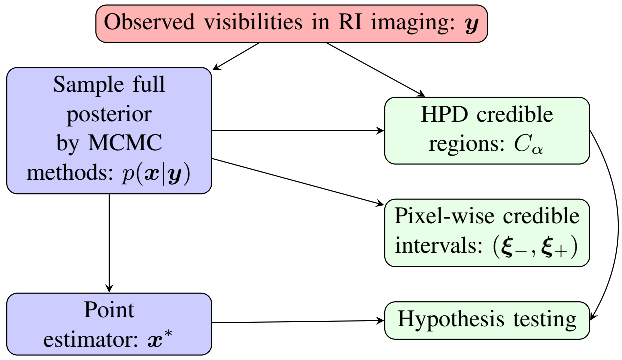

Fig. 1. UQ methodology I: procedure based on MCMC techniques [1].

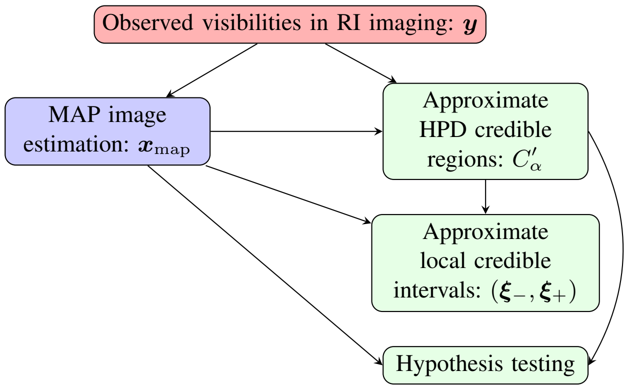

Fig. 2. UQ methodology II: procedure based on MAP estimation [2].

References

[1] X. Cai, M. Pereyra, and J. D. McEwen, "Uncertainty quantification for radio interferometric imaging I: proximal-MCMC methods," *MNRAS*, vol. 480, no. 3, pp. 4154–4169, 2018.

[2] ——, "Uncertainty quantification for radio interferometric imaging II: MAP estimation," *MNRAS*, vol. 480, no. 3, pp. 4170–4182, 2018.

[3] U. Rau, S. Bhatnagar, M. A. Voronkov, and T. J. Cornwell, "Advances in Calibration and Imaging Techniques in Radio Interferometry," *Proc. IEEE*, vol. 97, pp. 1472–1481, 2009.